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Dual-mixed variational principle in thermoelastodynamics – application to axially loaded beams

Balázs Tóth[1)]

[1)] *Institute of Applied Mechanics, University of Miskolc, Miskolc-Egyetemváros, Hungary*
[1)] mechtb@uni-miskolc.hu

ABSTRACT

A new four-field dual and mixed variational principle will be presented for linearly coupled thermoelastodynamic, one space-dimensional (1D) beam problems, using Lord-Shulman theory. In this variational formulation the entropy flux is introduced instead of the entropy as one of the independent variables. The fundamental variables of the variational principle are the axial displacement, momentum and normal stress, as well as the entropy flux. This variational principle is well-suitable for the transient analysis of 1D beam structures exposed to a large thermal shock of a short temporal domain. As an application of the mentioned variational formulation, a theoretical model will be presented for axially loaded 1D thermoelastic beams, serving as a theoretical basis for space-time *hp*-finite elements (*hp*-FEs).

1. INTRODUCTION

The complete system of algebraic and differential equation can be usually considered as the strong formulation for initial-boundary value problems (IBVP) of continuum thermodynamics. Nevertheless, these mathematical models can be cast in variational forms to permit numerical treatment in space-time domain. These variational principles always reformulate the original IBVP of continua in global forms. Usually these variational formulations are applied to construct FEs for discretization of space-time domain. Besides, the second sound effect has a great importance especially when the continua is exposed to a thermal shock load and/or very high heat fluxes which can be experienced for example in nuclear reactors and particle accelerators. In view of this, the aim of this paper is to present a four-field dual-mixed variational formulation which is well-suitable for the development of *hp*-version space-time FEs for 1D beam problems in linearly coupled, thermoelastodynamics with second sound effects, using Lord-Shulman theory.

[1)] Associate Professor

2. STRONG FORMULATION

It will be assumed that the thermoelastic beam of length L is exposed to the volumetric force density b in $x \in [0, L]$ and the traction force f per unit cross-sectional area at $x = x_\sigma$, as well as the axial displacement $\tilde{u}$ is prescribed at cross-sectional coordinate $x = x_u$. Besides, the internal heat generation r in $x \in [0, L]$ and the external heat flow $\tilde{q}$ through the cross-sectional area at $x = x_q$, as well as the temperature change $\tilde{\theta} = \tilde{T} - T_0$ is prescribed at the boundary $x = x_\theta$, where $\tilde{T}$ is the prescribed value of the absolute temperature and T_0 is the reference temperature. In the linearized theory of coupled thermoelastodynamics it will be supposed that the small temperature change for which $|\theta/T_0| \ll 1$ holds true, is simultaneously accompanied in the whole beam by small deformation (axial strain), as well as the material parameters do not vary significantly during the temperature increase (decrease), namely these are considered to be independent of θ.

2.1 Basic system of equation

The related basic system of equation can be grouped into (i) the basic algebraic and differential equations, as well as (ii) the initial and boundary conditions (IC and BC). The algebraic and differential forms of the basic equations for 1D, linearly coupled thermoelastodynamic beam problems associated with second sound effect are the entropy equation

$$T_0 \, \dot{s} = q' + r \, ,$$

where s stands for the volumetric entropy density and q is the heat flux per unit surface (or cross section area), as well as the prime in the superscript and the dot mean the time- (*t*) and space (*x*) derivative, respectively; the equation of motion

$$\sigma' + b = \dot{p} \, ,$$

in which σ is the axial normal stress and p is the axial momentum; the geometric equation

$$u' = \varepsilon \, ,$$

here ε is the axial strain; the kinematic equation

$$v = \dot{u} \, ,$$

where v is the axial velocity; the constitutive equations

$$v = \frac{p}{\rho} \, ,$$

$$\theta = \frac{T_0}{c_\sigma}(s - \alpha\sigma) \, ,$$

or

$$s = \alpha\sigma + \frac{c_\sigma}{T_0}\theta \, ,$$

the Duhamel-Neumann relation

$$\varepsilon = \left(\frac{1}{E} - \frac{T_0}{c_\sigma}\alpha^2\right)\sigma + \frac{T_0}{c_\sigma}\alpha s \, ,$$

or

$$\varepsilon = \frac{\sigma}{E} + \alpha\theta \, ,$$

as well as the modified Fourier's law with the heat flux rate according to the Lord-Shulman model (Lord and Shulman, 1967):

$$q + \tau \dot{q} = -k\theta',$$

where $c_\sigma, \alpha, k, \rho, \tau, E$ are the materials constants interpreted as the specific heat at constant stress, the thermal expansion coefficient, the heat conductivity coefficient, the material density, the relaxation time and the elasticity modulus. In order to eliminate the entropy and heat flux from the formulation above, the entropy flux h is introduced as

$$s = -h'$$

and

$$q = T_0\, \dot{h}$$

satisfying the homogeneous form of the entropy equation provided that $r = 0$ in what follows. Making use of the above two equations, the basic equations becomes

$$\sigma' + b = \dot{p}\ ,$$
$$u' = \varepsilon\ ,$$
$$v = \dot{u}\ ,$$
$$v = \frac{p}{\rho}\ ,$$
$$\theta = -\frac{T_0}{c_\sigma}(h' + \alpha\sigma)\ ,$$

or

$$h' = \alpha\sigma + \frac{c_\sigma}{T_0}\theta$$

and

$$\varepsilon = \left(\frac{1}{E} - \frac{T_0}{c_\sigma}\alpha^2\right)\sigma + \frac{T_0}{c_\sigma}\alpha h'\ ,$$

or

$$\varepsilon = \frac{\sigma}{E} + \alpha\theta\ ,$$

as well as

$$T_0\left(\dot{h} + \tau\ddot{h}\right) = -k\theta'.$$

2.2 Initial and boundary conditions

Two temporal prescriptions will be given not only for the entropy flux h and/or its rate $\dot{h}$ but also for the displacement u and/or the momentum p:

$$h(x, t_\lambda) = {}_\lambda h(x)\ ,$$
$$\dot{h}(x, t_\gamma) = {}_\gamma \dot{h}(x)$$

and

$$u(x, t_\lambda) = {}_\lambda u(x)\ ,$$
$$p(x, t_\gamma) = {}_\gamma p(x)\ ,$$

in which t_λ and/or t_γ are defined as the set of the time instants at which these quantities are prescribed by functions ${}_\lambda h(x)$ and/or ${}_\gamma \dot{h}(x)$, as well as ${}_\lambda u(x)$ and/or ${}_\gamma p(x)$. Here the Greek indices appearing in the subscripts are 1 and/or 2. Two-two of the four-four possible impositions have to be chosen. This selection process can be performed in six different ways. The BCs to the basic system of equation are given for the temperature change, the entropy flux, the axial displacement and the normal stress as

$$\theta = \tilde{\theta}(x_\theta, t)\,,$$
$$\dot{h} = \frac{\tilde{q}}{T_0}\left(x_q, t\right),$$
$$u = \tilde{u}(x_u, t)\,,$$
$$\sigma = f(x_\sigma, t)\,,$$

where x_θ, x_q, x_u and x_σ are the space coordinate of the cross section at which the temperature change, the entropy flux (through the heat flux) and the axial stress are prescribed.

3. VARIATIONAL FORM

The variational form for the linearly coupled thermoelasticity problems of 1D beam is much more advantageous to develop approximate solutions for the broad range of complex heat conduction problems for example with the application of space-time finite element method.

The applied four-field dual and mixed weak form of the coupled thermoelastodynamic, 1D, beam problems associated with second sound effect reads: find the trial functions σ, h, u, θ satisfying the BCs for the axial stress and the entropy flux, as well as the related ICs in view of the appropriately selected expressions from Table 1 in Tóth (2018) and Table 1-2 in Tóth (2016) such that

$$\begin{aligned}
&\delta F(\sigma, h, u, \theta, \delta\sigma, \delta h, \delta u, \delta\theta) \\
&= \int_{t_1}^{t_2}\int_0^L \left(\frac{T_0}{k}\dot{h}\,\delta h - T_0\tau\,\dot{h}\,\delta\dot{h} - \frac{\sigma}{E}\delta\sigma - \alpha\,\theta\,\delta\sigma - \alpha\,\sigma\,\delta\theta\right)dxdt \\
&- \int_{t_1}^{t_2}\int_0^L \left(\frac{c_\sigma}{T_0}\theta\,\delta\theta - \rho\,\dot{u}\,\delta\dot{u} + \sigma'\delta u + u\,\delta\sigma' + b\,\delta u + h'\delta\theta + \theta\,\delta h'\right)dxdt \\
&+ \int_{t_1}^{t_2}\left(\int_{x_u}\tilde{u}\,\delta\sigma\,dA + \int_{x_\theta}\tilde{\theta}\,\delta h\,dA\right)dt + \int_0^L(\delta A + \delta C)\,dx = 0
\end{aligned}$$

for all test functions $\delta\sigma, \delta h, \delta u, \delta\theta$ satisfying the homogeneous form of the BCs for the axial stress and the entropy flux, as well as the associated ICs in view of the properly chosen expressions for $\delta A, \delta C$ from Table 1 in Tóth (2018) and Table 1-2 in Tóth (2016).

In the form of the four-field dual and mixed variational principle presented above, the solution of the linearly coupled thermoelasticity problems associated with second sound effect can be characterized as the unique stationary point of functional

$$F(\sigma, h, u, \theta) = \int_{t_1}^{t_2}\int_0^L \left(\frac{T_0}{2k}\dot{h}^2 - \frac{T_0\tau}{2}\dot{h}^2 - \frac{\sigma^2}{2E} - \alpha\theta\sigma - \frac{c_\sigma}{2T_0}\theta^2 + \frac{\rho}{2}\,\dot{u}^2\right)dxdt$$

$$- \int_{t_1}^{t_2}\int_{0}^{L} [(\sigma' + b)\, u + h'\theta]\, dxdt + \int_{t_1}^{t_2} \left(\int_{x_u} \tilde{u}\, \sigma\, dA + \int_{x_\theta} \tilde{\theta}\, h\, dA \right) dt + \int_{0}^{L} (A + C)\, dx = 0$$

for the independent variables σ, h, u, θ which satisfy the kinematic equation, the first three constitutive equations and the BCs for the axial stress and the entropy flux, as well as the corresponding ICs.

4. SUMMARY

As a conclusion, in the form of the four-field dual-mixed variational formulation presented previously, (i) the BCs for the axial normal stress and the entropy flux are essential, while the displacement- and temperature BC come naturally from the variational principle, additionally (ii) the variational integrals A and C are responsible for the appropriate treatment of the ICs. As in the considered problem, the internal mechanical processes are accompanied by energy dissipation, the first variation of the action functional F cannot be expressed explicitly. However, using the concept of Biot-Rayleigh for viscous damping, the irreversible processes can be described energetically by introducing a scalar dissipation function to the action functional (Biot, 1954 and Rayleigh, 1887). Here the variation of the dissipation function is interpreted ad hoc, see the details for example in Tóth (2018). This standard method was used to modelling a broad range of dissipative systems.
The variational principle presented can serve a promising mathematical basis for the development of new, reliable and efficient mixed space-time *hp*-FEs and numerical time integration schemes suitable for modelling transient thermal stress analyses of 1D beam structures that are exposed to a thermal shock load within a very short time interval (for example ramp- and/or step-type external heat fluxes).

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