The 2015 World Congress on
Advances in Structural Engineering and Mechanics (ASEM15)
Incheon, Korea, August 25-29, 2015

Bending of a circular plate with stiff edge lying on a soft functionally-graded elastic layer under the action of arbitrary axisymmetric loading

Sergey S. Volkov[1)], Alexander N. Litvinenko[2)], Sergey M. Aizikovich[3)], Yun-Che Wang[4)] and Andrey S. Vasiliev[5)]

[1), 3), 5)] *Research and Education Center "Materials", Don State Technical University, Rostov-on-Don 344000, Russia*
[2)] *Institute of Mathematics, Mechanics and Computer Science, Southern Federal University, Rostov-on-Don 344090, Russia*
[4)] *Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan*
[3)] *saizikovich@gmail.com*

ABSTRACT

A circular plate with constant thickness, finite radius and stiff edge lying on an elastic half-space is considered. The half-space consists of soft functionally graded layer (coating) with arbitrary varying elastic properties and homogeneous elastic substrate. The plate bends under the action of arbitrary axisymmetric distributed load and response from the elastic half-space. A semi-analytical solution for the problem effective in whole range of geometric (relative layer thickness) and mechanical (elastic properties of coating and substrate, stiffness of the plate) properties is constructed using coupled asymptotic method [1]. Approximated analytical expressions for the contact stresses and deflections of the plate are provided. Numerical results showing the qualitative dependence of the solution from the initial parameters of the problem are obtained with high precision.

1. INTRODUCTION

The problem of plate bending on an isotropic homogeneous elastic foundation was considered in [2]. The solution was constructed by providing the contact stresses in the form of a power series with subsequent determination of the coefficients of expansion from infinite algebraic equations. The similar problem was also solved using collocation method [3] and approach based on approximations of the kernels by orthogonal polynomials [4], [5]. The convergence of the solution to the exact one was not

[1), 5)] Ph.D., Senior Researcher
[2), 4)] Ph.D., Associate Professor
[3)] D.Sc., Ph.D., Head of Laboratory

investigated. Singular and regular asymptotic methods were used to construct solutions effective for big or small values of the geometric parameter of the problem [4], [6].

Most of the known solutions are applicable only for rigid plates. And very few, particularly those described in [4], [5], [6] are applicable either for flexible or rigid plates. There is a number of recent investigations in this area [7], [8].

In this work we describe an approach, based on the coupled asymptotic method [1] of solving a certain type of dual integral equations, which allows one to construct analytical solution of the problem in unified form, applicable for any values of geometric and mechanical properties.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Circular plate of radius *R* and constant thickness *h* lying on the boundary of an elastic half-space, consisting of inhomogeneous layer (coating) with thickness H and homogeneous half-space (substrate). We use a cylindrical coordinate system *r*, φ, *z*, where z axis is perpendicular to the surface of the coating and passes through the center of the plate. Plate is bent under the action of an axisymmetric distributed load *p**(*r*) and response from the layer.

Young's modulus E and Poisson's ratio ν of the foundation vary with depth according to the following

$$\mathrm{E}(z),\nu(z)=\begin{cases}E_1(z)\,,\nu_1(z)\,,-H\le z\le 0\\ E_2,\nu_2,\qquad -\infty<z<-H\end{cases},\ E_2,\nu_2=\mathrm{const} \tag{1}$$

where $E_1(z)$, $\nu_1(z)$ are arbitrary continuously differentiable functions. Hereafter, indexes $_1$ and $_2$ correspond to the coating and to the substrate, respectively.

The layer and the substrate are assumed to be glued without sliding:

$$z=-H:\tau_{zr}^1=\tau_{zr}^2,\ \sigma_z^1=\sigma_z^2,\ w^1=w^2,\ \ u^1=u^2 \tag{2}$$

Outside of the punch, the surface is traction-free:

$$z=0:\ \tau_{zr}^1=0,\ \begin{cases}\sigma_z^1=0,\quad r>R\\ w^1=-w^*(r),\ r\le R\end{cases} \tag{3}$$

The stresses and the displacements vanish at *r*→∞ and *z*→∞.

The quantities of primary interest are the contact stresses under the plate $q^*(r)=\sigma_z\big|_{z=0}$, the deflections of the plate $w^*(r)$, radial and tangential torques M_r, M_φ.

We consider two types of the boundary conditions on the edges of the plate:

a) Plate with a stiff edge: $\left.\dfrac{\partial w^*}{\partial r}\right|_{r=R}=0,\ \left.\dfrac{\partial}{\partial r}\left(\Delta w^*\right)\right|_{r=R}=0$ (4)

Such boundary condition arises in the calculation of tank bottoms, sunk wells, etc. having quite stiff contours.

b) Free edge:

$$\left.\left(\frac{\partial^2 w^*}{\partial r^2}+\frac{\nu_{\mathrm{plate}}}{r}\frac{\partial w^*}{\partial r}\right)\right|_{r=R}=0,\ \ \left.\frac{\partial}{\partial r}\left(\Delta w^*\right)\right|_{r=R}=0 \tag{5}$$

Boundary conditions (5) arise in modeling foundations, free-lying plates, etc.

We use the following notations: ν_{plate} is the Poisson's ratio of the plate, $\Delta = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$ is the Laplace operator.

3. SOLUTION OF THE PROBLEM

3.1 Dual integral equation

To reduce the problem to the solution of the dual integral equation we use the classical approach based on the Hankel's integral transformations technique:

$$w^1(r,z) = \int_0^\infty W_1(\gamma,z)\,\mathrm{J}_0(\gamma r)\gamma d\gamma\,, \tag{6}$$

$$Q^*(\gamma) = \int_0^R q^*(\rho)\,\mathrm{J}_0(\gamma\rho)\rho d\rho\,,\;\; q^*(r) = \int_0^\infty \gamma Q^*(\gamma)\mathrm{J}_0(r\gamma)d\gamma \tag{7}$$

Then (3) leads to the following:

$$\int_0^\infty W_1\,(\gamma,0)\gamma\,\mathrm{J}_0(\gamma r)d\gamma = -w^*(r),\;\; r \le R \tag{8}$$

Let us introduce the following notations:

$$W_1^*(\gamma,z) = -\Theta\frac{\gamma W_1(\gamma,z)}{Q^*(\gamma)}\,,\;\; \Theta = \frac{E(0)}{2(1-\nu^2(0))} \tag{9}$$

Using (9) we rewrite equation (8) in the form:

$$\int_0^\infty W_1^*(\gamma,0)Q^*(\gamma)\,\mathrm{J}_0(\gamma r)d\gamma = \Theta w^*(r),\;\; r \le R \tag{10}$$

Substituting (7) into (10) we get:

$$\int_0^R q^*(\rho)\rho d\rho\int_0^\infty W_1^*(\gamma,0)\,\mathrm{J}_0(\gamma r)\,\mathrm{J}_0(\gamma\rho)d\gamma = \Theta w^*(r),\;\; r \le R \tag{11}$$

Let us introduce the dimensionless variables and functions:

$$\gamma H = u,\, \lambda = H/R,\, r' = r/R,\, \rho' = \rho/R,\, W_1^*(u/H,0) = L(u),\, q^*(r) = q(r')DR^{-3},$$
$$z' = z/H,\, w^*(r) = w(r')R,\; p^*(r) = p(r')DR^{-3},\, \alpha = u\lambda^{-1},\, s = \Theta R^3/D$$

where *D* is cylindrical stiffness of the plate, parameter *s* is dimensionless bending stiffness of the plate. Function *L*(*u*) is the kernel transform of the integral equation, independent of the applied loading *p**(*r*) and characterizes the compliance of the elastic foundation. The kernel transform *L*(*u*) is equal to that appearing in the contact problem of the indentation of a rigid stamp [9].

Omitting the primes in (11) we get the Fredholm integral equation of the first king over the function *q*(*ρ*)

$$\int_0^1 q(\rho)\rho d\rho\int_0^\infty L(\lambda\alpha)\,\mathrm{J}_0(\alpha r)\,\mathrm{J}_0(\alpha\rho)d\alpha = sw(r),\;\; r \le 1 \tag{12}$$

Equation (12) is equivalent to the following dual integral equation:

$$\begin{cases} \int_0^\infty Q(\alpha)L(\alpha\lambda)\,\mathrm{J}_0(\alpha r)d\alpha = sw(r),\ 0 \le r \le 1 \\ \int_0^\infty Q(\alpha)\,\mathrm{J}_0(\alpha r)d\alpha = 0,\ r > 1 \end{cases} \tag{13}$$

According to the Kirchhoff's plate model the deflection of the plate *w*(*r*) has to satisfy the differential equation of bending of the plate:

$$\mathbf{L_0}w(r) = p(r) - q(r),\ \mathbf{L_0} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 0 \le r \le 1, \tag{14}$$

3.2 Solution of the problem

In the present paper we construct the solution only for a case of a plate with a stiff edge (see (4)).

We represent the deflection function in terms of series with respect to eigenfunctions of oscillations of a circular plate with free edges (similar to Tseitlin [10]):

$$w(r) = \sum_{m=0}^{\infty} w_m \varphi_m(r), \quad w_m = \int_0^1 w(\rho)\varphi_m(\rho)\rho d\rho, \quad \varphi_m(r) = A_m\,\mathrm{J}_0(k_m r) \tag{15}$$

where constants k_m are the roots of the equation $\mathrm{J}_1(k_m) = 0$

and $A_m = \sqrt{2}/\mathrm{J}_0(k_m), m = 1,2,\ldots.$

Due to the linearity of the problem the contact stresses *q*(*r*) and it's Hankel transform *Q*(*α*) can be represented as:

$$q(r) = \sum_{m=0}^{\infty} w_m q_m(r),\ Q(\alpha) = \sum_{m=0}^{\infty} w_m Q_m(\alpha)\ \ 0 \le r \le 1, \tag{16}$$

Substituting (15) and (16) into (13) we get the dual integral equation over the function $Q_m(\alpha)$:

$$\begin{cases} \int_0^\infty Q_m(\alpha)L(\lambda\alpha)\,\mathrm{J}_0(r\alpha)d\alpha = s\varphi_m(r),\ r \le 1 \\ \int_0^\infty Q_m(\alpha)\,\mathrm{J}_0(r\alpha)\alpha d\alpha = 0, \qquad r > 1 \end{cases} \tag{17}$$

Let us apply to the first equation in (17) the integral operator $U_1^t\varphi = \frac{d}{dt}\int_0^t \frac{r\varphi(r)dr}{\sqrt{t^2 - r^2}}$ while to the second equation in (17) we will apply the integral operator $U_2^t\varphi = \int_0^\infty \frac{r\varphi(r)dr}{\sqrt{r^2 - t^2}}$. As the result of that we obtain

$$\begin{cases} \int_0^\infty Q_m(\alpha)L(\lambda\alpha)\cos(t\alpha)d\alpha = g_m(t), & t \le 1 \\ \int_0^\infty Q_m(\alpha)\cos(t\alpha)d\alpha = 0, & t > 1 \end{cases} \quad (18)$$

We used the following notation:

$$g_m(t) = sU_1^t\varphi_m = s\frac{d}{dt}\int_0^t \frac{r\varphi_m(r)dr}{\sqrt{t^2 - r^2}} = sA_m\cos(k_m t) \quad (19)$$

In general, the kernel transform *L*(*α*) depends on the properties of the nonhomogeneous materials. In [11] it was shown that *L*(*α*) possesses the following properties:

$$L(\gamma) = A + B\gamma + C\gamma^2 + O(\gamma^3), \gamma \to 0$$
$$L(\gamma) = 1 + D\gamma^{-1} + E\gamma^{-2} + O(\gamma^{-3}), \gamma \to \infty \quad (20)$$

where *A*, *B*, *C* and *D* are certain constants which values depend on the material properties. It was shown [12], [13] that the kernel transform *L*(*α*) with high precision can be approximated by the expression:

$$L(\alpha\lambda) \approx L_N(\alpha\lambda) = \frac{P_1(\alpha^2\lambda^2)}{P_2(\alpha^2\lambda^2)}$$
$$P_1(\alpha^2\lambda^2) = \prod_{i=1}^N (\alpha^2\lambda^2 + a_i^2), \quad P_2(\alpha^2\lambda^2) = \prod_{i=1}^N (\alpha^2\lambda^2 + b_i^2) \quad (21)$$

where a_i and b_i are certain constants obtained by approximation the kernel transform *L*(*α*). A detailed description of the process of determining coefficients $a_i, b_i (i = 1..N)$ is described in [13].

It was also shown [1] that the solution of approximate dual integral equation resulting from replacing transform *L*(*α*) in (18) by its approximation $L_N(\alpha)$ is asymptotically exact for both thin and thick coatings, i.e. for $\lambda \to 0$ and $\lambda \to \infty$.

Let us introduce the functions:

$$d_m(t) = \frac{1}{sA_m}\int_0^\infty Q_m(\alpha)\cos(t\alpha)d\alpha \quad (22)$$

then using operational analysis [14] and equations (21),(22), we represent first equation in (18) in operator form:

$$P_1(-D)d_m(t) = P_2(-D)g_m(t), \quad D = \frac{d^2}{dt^2}, \quad t \in [0,1] \quad (23)$$

The solution of the differential equation (23) for d_m has the form:

$$d_m(t) = \sum_{i=1}^N \vartheta(C_i^m, D_i^m, a_i\lambda^{-1}t) + L_N^{-1}(\lambda k_m)\cos(k_m t) \quad (24)$$

where coefficients C_i^m and D_i^m are arbitrary constants, $\vartheta(x, y, z) = x\,\mathrm{ch}(z) + y\,\mathrm{sh}(z)$.

Using expressions (22) and (24) equations (18) can be rewritten as follows

$$\begin{cases} \dfrac{1}{sA_m}\displaystyle\int_0^\infty Q_m(\gamma)\cos(t\alpha)d\alpha = \sum_{i=1}^{N} \vartheta(C_i^m, D_i^m, a_i\lambda^{-1}t) + L_N^{-1}(\lambda k_m)\cos(k_m t), \quad 0 \le t \le 1 \\ \displaystyle\int_0^\infty Q_m(\gamma)\cos(t\alpha)d\alpha = 0, \quad t > 1 \end{cases} \tag{25}$$

Inverting equations (25) using the Fourier's transform we obtain:

$$Q_m(\alpha) = \frac{2sA_m}{\pi}\left(\sum_{i=1}^{N}\left[\frac{\vartheta(C_i^m, D_i^m, \frac{a_i}{\lambda})\alpha\sin\alpha + \frac{a_i}{\lambda}\vartheta(D_i^m, C_i^m, \frac{a_i}{\lambda})\cos\alpha}{\alpha^2 + A_i^2\lambda^{-2}} - \frac{a_i\lambda^{-1}D_i^m}{\alpha^2 + a_i^2\lambda^{-2}}\right] + \right. \tag{26}$$

$$\left. + L_N^{-1}(\lambda k_m)\frac{\alpha\sin(\alpha)\cos(k_m) - k_m\sin(k_m)\cos(\alpha)}{\alpha^2 - k_m^2}\right)$$

Inverting the Hankel transform formula (26) and using Parseval's identity gives:

$$\begin{aligned} q_0(r) &= \frac{2\sqrt{2}}{\pi}s\left[\frac{1}{L_N(0)\sqrt{1-r^2}} + \sum_{i=0}^{N}\Psi(r, a_i\lambda^{-1}, C_i^0, D_i^0)\right], \\ q_m(r) &= \frac{2}{\pi}A_m s\left[\frac{\Phi(r, ik_m)}{L_N(\lambda k_m)} + \sum_{i=0}^{N}\Psi(r, a_i\lambda^{-1}, C_i^m, D_i^m)\right], \ m = 1,2,\ldots \end{aligned} \tag{27}$$

where:

$$\Psi(r, A, C, D) = \frac{\vartheta(C, D, A)}{\sqrt{1-r^2}} - CA\int_r^1 \frac{\mathrm{sh}(At)dt}{\sqrt{t^2 - r^2}} - AD\int_r^1 \frac{\mathrm{ch}(At)dt}{\sqrt{t^2 - r^2}}$$

$$\Phi(r, A) = \frac{\mathrm{ch}\,A}{\sqrt{1-r^2}} - A\int_r^1 \frac{\mathrm{sh}\,At dt}{\sqrt{t^2 - r^2}}$$

To determine constants C_i, D_i we substitute the expressions (27) into equation (18). The set of constants $C_i^m (m = 0,1,2,...; i = 1..N)$ is determined from the system of linear algebraic equations below, while $D_i^m = 0 (\forall m \forall i)$.

$$\begin{aligned} &\sum_{i=1}^{N} C_i^0 \eta(a_i\lambda^{-1}, b_k\lambda^{-1}) + L_N^{-1}(0)\lambda b_k^{-1} = 0, \quad k = 1,2,\ldots,N \\ &\sum_{i=1}^{N} C_i^m \eta(a_i\lambda^{-1}, b_k\lambda^{-1}) + \frac{\eta(ik_m, b_k\lambda^{-1})}{L_N(\lambda k_m)} = 0, \ k = 1,2,\ldots,N, \quad m = 1,2,\ldots \end{aligned} \tag{28}$$

where

$$\eta(x, y) = \frac{x\,\mathrm{sh}\,x + y\,\mathrm{ch}\,x}{y^2 - x^2}$$

Contact stresses $q_m(r)$ and pressure applied to the plate $p(r)$ can be represented as a following series:

$$q_m(r) = \sum_{j=0}^{\infty} y_j^m \varphi_j(r), \quad y_j^m = \int_0^1 q_m^N(\rho)\varphi_j(\rho)\rho d\rho \tag{29}$$

$$p(r) = \sum_{m=0}^{\infty} p_m \varphi_m(r), \quad p_m = \int_0^1 p(\rho)\varphi_m(\rho)\rho d\rho \tag{30}$$

Substituting (15), (16), (29) and (30) into (14) we get the infinite system of a linear algebraic equations over w_m:

$$w_m + k_m^{-4}\sum_{j=0}^{\infty} w_j E_j^m = p_m k_m^{-4} \quad m = 0,1,2,\ldots; \tag{31}$$

where:

$$\mathrm{E}_j^m = 2\pi^{-1}A_j s A_m\left(L_N^{-1}(\lambda k_j)\xi(k_j,k_m) + \sum_{n=0}^{N} C_n^m \zeta(a_n\lambda^{-1},k_m)\right), \quad j = 1,2,..., m = 0,1,2,...$$

$$\xi(x,y) = \frac{\cos x \sin y}{y} + \frac{x}{2y}\left(\frac{\sin(x-y)}{x-y} - \frac{\sin(x+y)}{x+y}\right)$$

$$\zeta(x,y) = \frac{x \,\mathrm{sh}\, x \cos y + y \sin y \,\mathrm{ch}\, x}{x^2 + y^2}$$

In particular:

$$E_0^m = 2\pi^{-1}\sqrt{2} s A_m\left[L_N^{-1}(0)k_m^{-1}\sin k_m + \sum_{n=1}^{N} C_n^0 \zeta(a_n\lambda^{-1},k_m)\right],$$

$$E_0^0 = \pi^{-1}4s\left[L_N^{-1}(0) + \lambda\sum_{n=1}^{N} C_n^0 a_n^{-1} sh(a_n\lambda^{-1})\right],$$

$$E_j^0 = 2\pi^{-1}\sqrt{2}A_j s\left[L_N^{-1}(\lambda k_j)k_j^{-1}\sin k_j + \lambda\sum_{n=1}^{N} C_n^j a_n^{-1} sh(a_n\lambda^{-1})\right]$$

Using reduction method to the infinite system (31) we get the finite system:

$$w_m k_m^4 + \sum_{j=0}^{K} w_j \mathrm{E}_j^m = p_m, \quad m = 0, 1, 2, \ldots, K \tag{32}$$

After determining parameters w_m $(m = 0,1,\ldots,K)$ for a fixed value K and substituting them into the expression (16) we finally get the contact stresses $q(r)$ and from the expression (15) we get the deflections of the plate $w(r)$.
Radial torque M_r and tangential torque M_φ of the plate can be represented in the following expressions over the deflections:

$$M_r = \left(-\frac{D}{R^2}\right)\left(\frac{d^2w}{dr^2} + \frac{\nu_{plate}}{r}\frac{dw}{dr}\right), \; M_\varphi = \left(-\frac{D}{R^2}\right)\left(\nu_{plate}\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) \tag{33}$$

where D is the is cylindrical stiffness of the plate.

Using the expression (15) and (33) we get the expressions for the torques:

$$M_r = \frac{D}{R}\sum_{m=0}^{K} w_m A_m k_m^2 \left[\frac{\nu_{\text{plate}} - 1}{k_m r} J_1(k_m r) + J_0(k_m r) \right],$$
$$M_\varphi = \frac{D}{R}\sum_{m=0}^{K} w_m A_m k_m^2 \left[\frac{1-\nu_{\text{plate}}}{k_m r} J_1(k_m r) + \nu_{\text{plate}} J_0(k_m r) \right], \quad (34)$$

For case of the plate with free edge defining by (5) the solution of the problem is presented in [15], but the expression contains a misprint in expressions (2.15, 2.16). Here we present the corrected expressions:

$$q_0^N(r) = \frac{2\sqrt{2}}{\pi} s \left[\frac{1}{L_N(0)\sqrt{1-r^2}} + \sum_{i=0}^{N} C_i^m \Phi(r, a_i \lambda^{-1}) \right] \quad (35)$$

$$q_m^N(r) = \frac{2}{\pi} A_m s \left[\frac{\Phi(r, ik_m)}{L_N(\lambda k_m)} - \frac{\mathrm{J}_1(k_m)}{\mathrm{I}_1(k_m)} \frac{\Phi(r, k_m)}{L_N(i\lambda k_m)} + \sum_{i=0}^{N} C_i^m \Phi(r, a_i \lambda^{-1}) \right], \; m = 1,2,\ldots \quad (36)$$

The way of determining the parameters C_i^m are similar to that we used in the present paper: we substitute the expressions (35), (36) into the dual integral equation (18) and get the system of linear algebraic equations. Values of parameter A_m and k_m corresponding to the plate with free edge are presented in [10].

4. NUMERICAL RESULTS

4.1 Soft homogeneous layer

Let us consider the bending of the plate lying on a layer with constant elastic moduli (E_1=*const*, ν_1=*const*) under the action of a uniformly distributed unit load: $p(r)=1$, $r<1$. We assume that the layer is much softer than the substrate ($E_2 \gg E_1(H)$) and use parameter $\beta = E_2/E_1(-H)$ to characterize softness of the layer. Let us consider case of a coating much softer than the substrate with β=100. The numerical results provided for a case of the plate with stiff edge (see (4)).

According to Gorbunov-Posadov [2] the plate assumed to be flexible (or of infinite radius) if s>10 and stiff if s<1. Numerical examples below are provided for three values of parameter s: s=100, s=0.01 and s=5.

Figure 1 contains graphs of the contact pressure under the plate of intermediate flexibility (s=5) lying on coatings of large or intermediate thickness λ. It is seen that the pressure on thick coatings (for instance, λ=10) are minimal under the center of the plate (*r*=0) and monotonically increases when *r* approaching the edge of the plate. For intermediate thickness of the coating (0.1<λ<1) the pressure under the center increases while the minimum value is reached near the edge of the plate (*r*=0.7..0.9).

The pressure distribution for small values of relative layer thickness λ has complicated nonmonotonic behavior (see Figure 2): near the end points of the contact region one can observe an increase and decrease in pressure, i.e. pressure spikes. With decrease in the relative thickness of the coating λ these pressure maxima and minima become higher, occupy less space, and move toward the end points of the contacts. Relative deflections ($w_{rel}(r) = w(r)/w(0)$) of the plate corresponding to the pressure provided on figure 1 and 2 are illustrated on the figure 3.

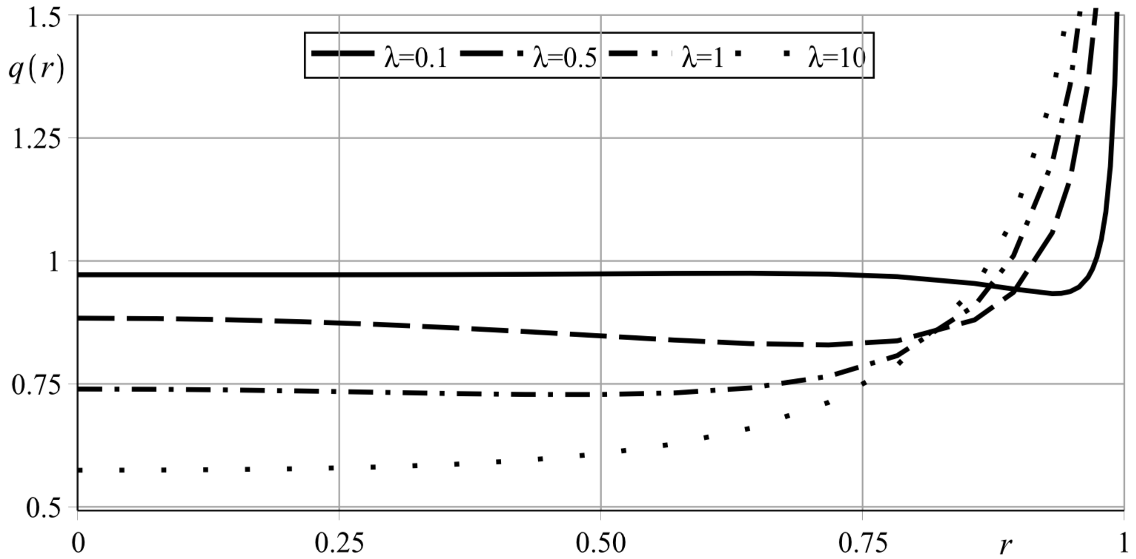

Figure 1. Graphs of pressure distribution *q*(*r*) versus *r*. The graphs are presented for soft coatings with β = 100, plate of intermediate flexibility s=5 and some large and intermediate coating thicknesses λ.

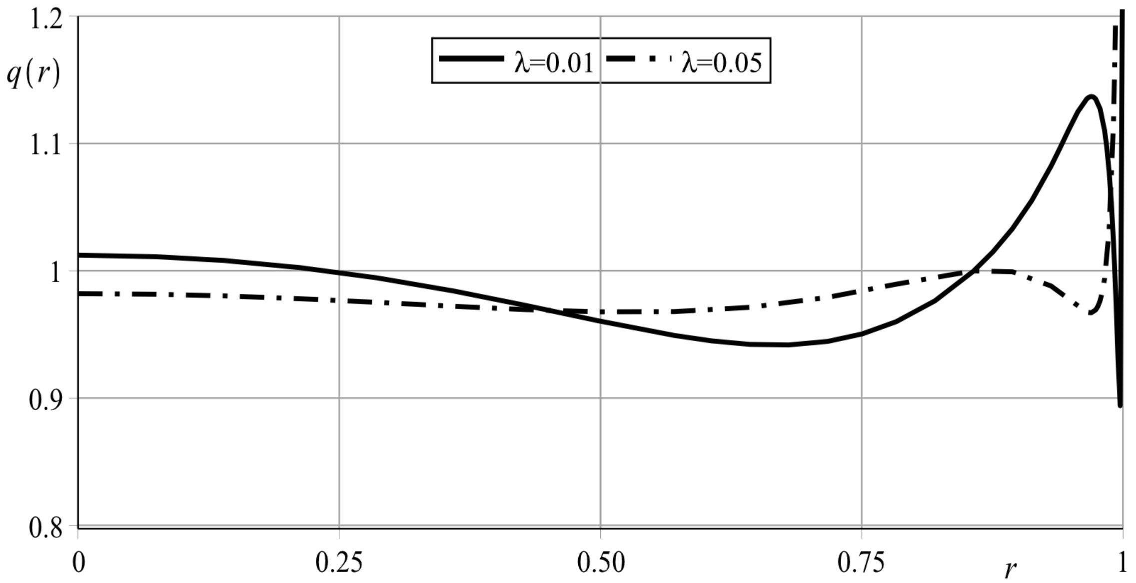

Figure 2. Graphs of pressure distribution *q*(*r*) versus *r*. The graphs are presented for soft coatings with β = 100, plate of intermediate flexibility s = 5 and some small coating thicknesses λ.

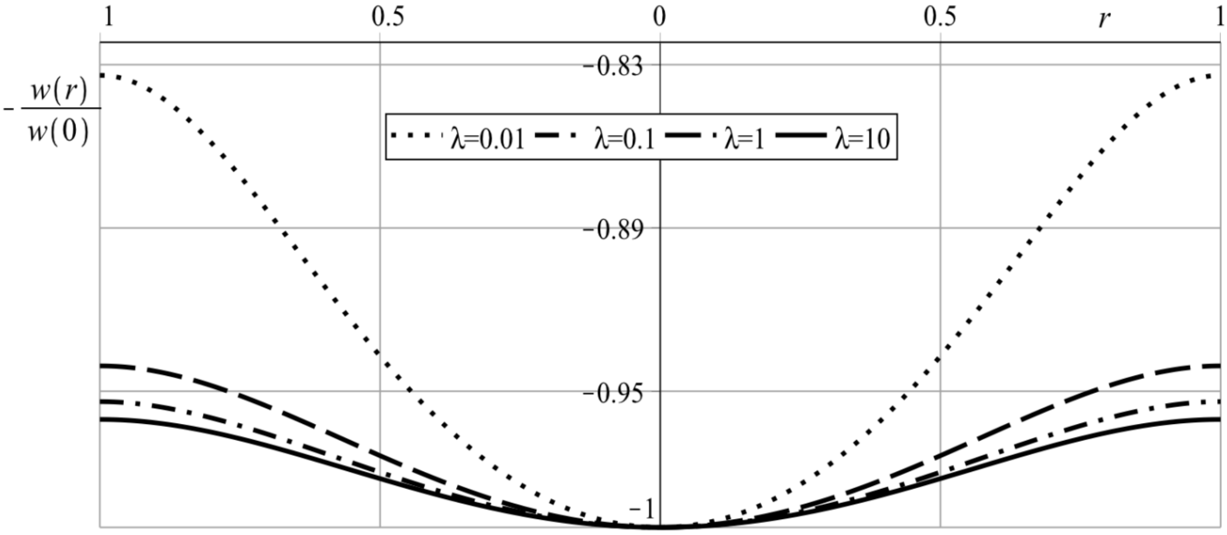

Figure 3. Graphs of relative deflections of the plate $w_{rel}(r)$ versus r. The graphs are presented for soft coatings with β = 100, plate of intermediate flexibility s = 5.

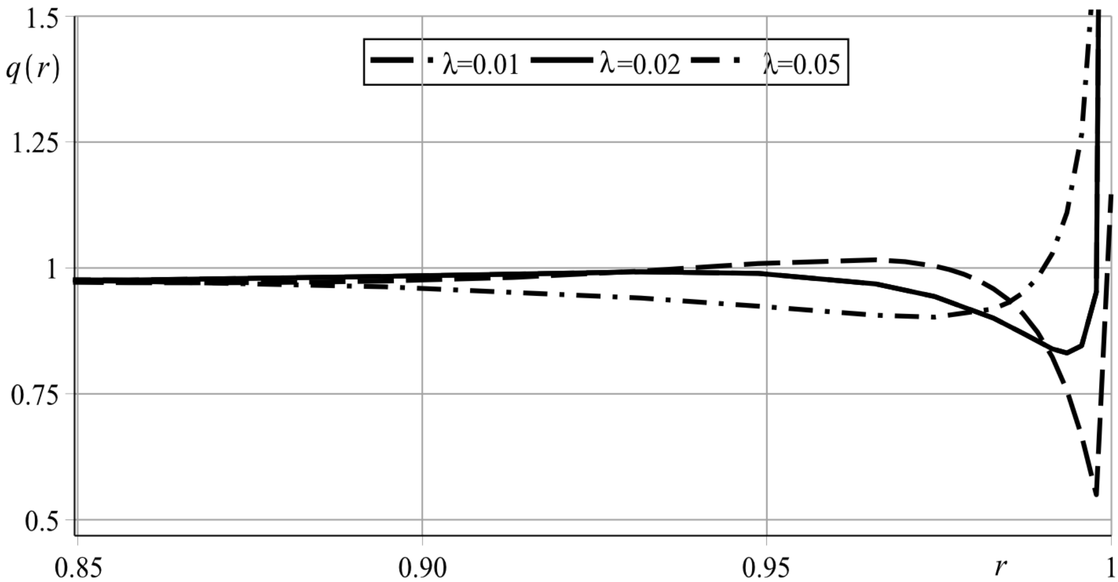

Figure 4. Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft coatings with β = 100, flexible plate with s = 100 and some small coating thicknesses λ.

For flexible plates the pressure under the center increases while near the edge its values sufficiently decrease (see figure 4), for stiff plates the opposite situation is observed (see figure 5).

To illustrate how the layer thickness influences the pressure we provide a dependence of the values $q(0)$ and $q(0.5)$ from the relative layer thickness λ (see figure 6). The graphs are provided for β=2, 5, 10, 100 and s=5. The pressure has local maximum for λ=(0.1..0.5). For λ>4 the pressure almost doesn't depend on the value of β. For λ<0.005 the pressure practically does not change with decreasing λ (the coating is so thin that practically has no effect on the pressure redistribution).

The pressure for the plates with free and stiff edges (see (5) and (4)) are illustrated in figure 7. It is seen that the pressure for the plate with free boundaries are greater in the neibourhood of *r*=0 and smaller near its edge than for the plate with stiff edge. The qualitative differences were not founded.

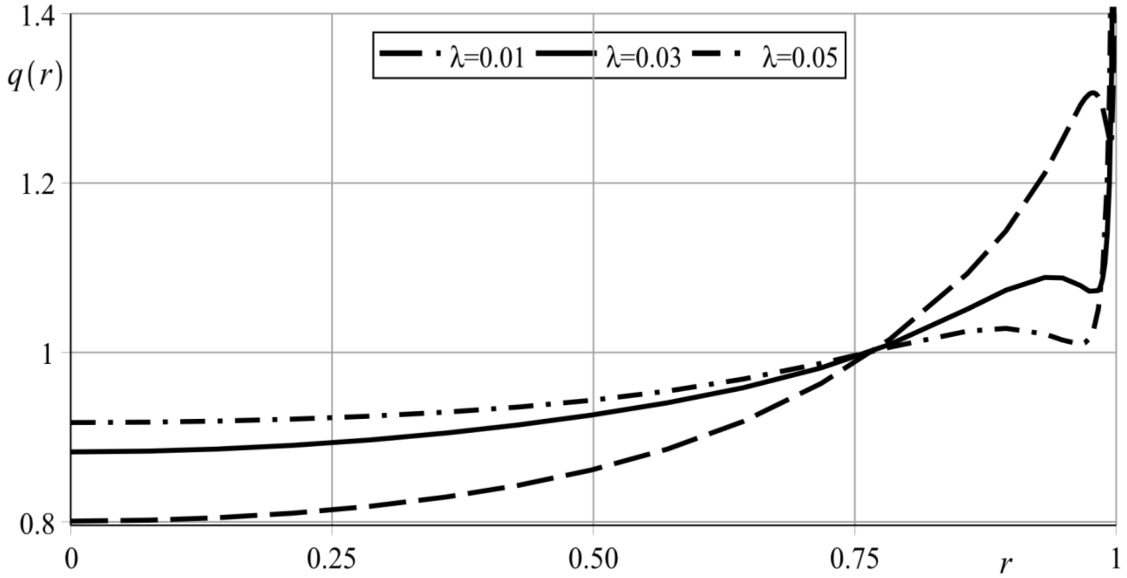

Figure 5. Graphs of pressure distribution *q*(*r*) versus *r*. The graphs are presented for soft coatings with β = 100, stiff plate with s = 0.01 and some small coating thicknesses λ.

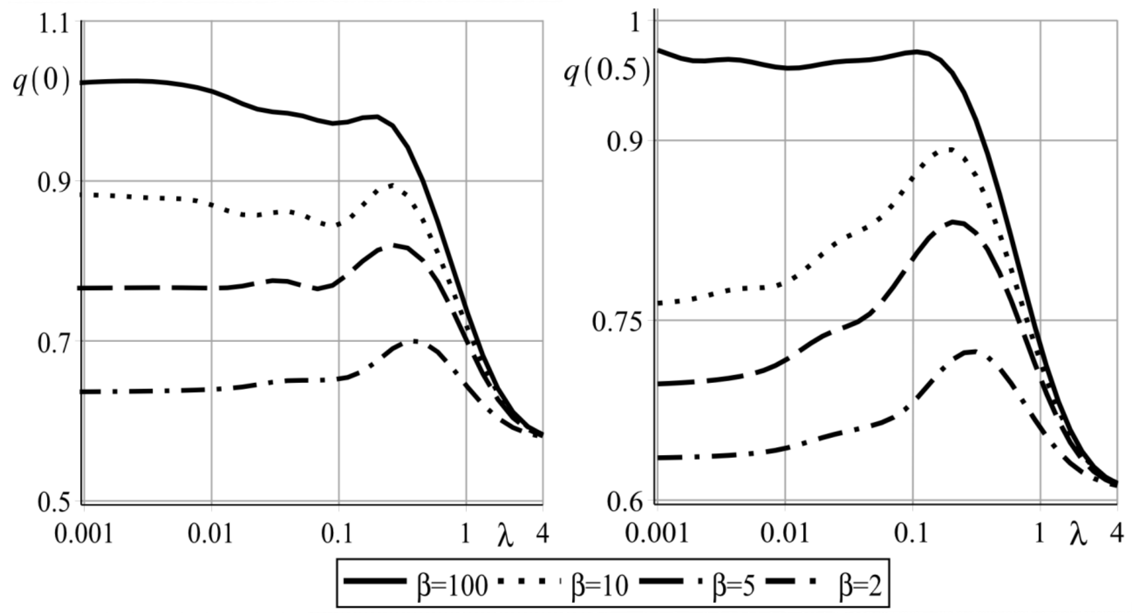

Figure 6. Graphs of pressure *q*(0) and *q*(0.5) versus λ. The graphs are presented for soft coatings with β = 2, 5, 10, 100, plate of intermediate flexibility s = 5.

Let the Young's modulus of the coating varies with depth according to one of the following rules:

coating 1: $E_1(z) = \frac{4.5}{7} + \frac{2.5}{7}\cos\left(2\pi\frac{z}{H}\right)$ coating 2: $E_1(z) = \frac{4.5}{2} - \frac{2.5}{2}\cos\left(2\pi\frac{z}{H}\right)$

The pressure under the center of the plate *q*(0) versus λ and pressure distribution versus *r* for functionally-graded coatings 1 and 2 and homogeneous coating for β=100 and s=5 are presented in the figures 8, 9. It is seen that the functionally graded properties of the coating sufficiently redistribure the pressure expecially for λ=1..10.

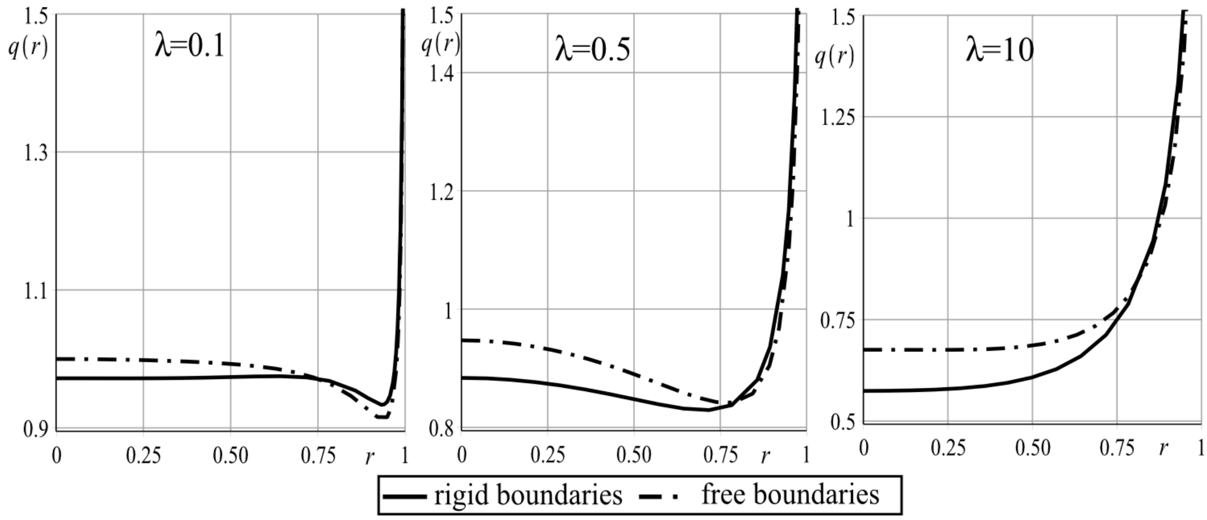

Figure 7. Graphs of pressure distribution *q*(*r*) versus *r*. The graphs are presented for soft coatings with β = 100, plate of intermediate flexibility s = 5 and some small coating thicknesses λ.

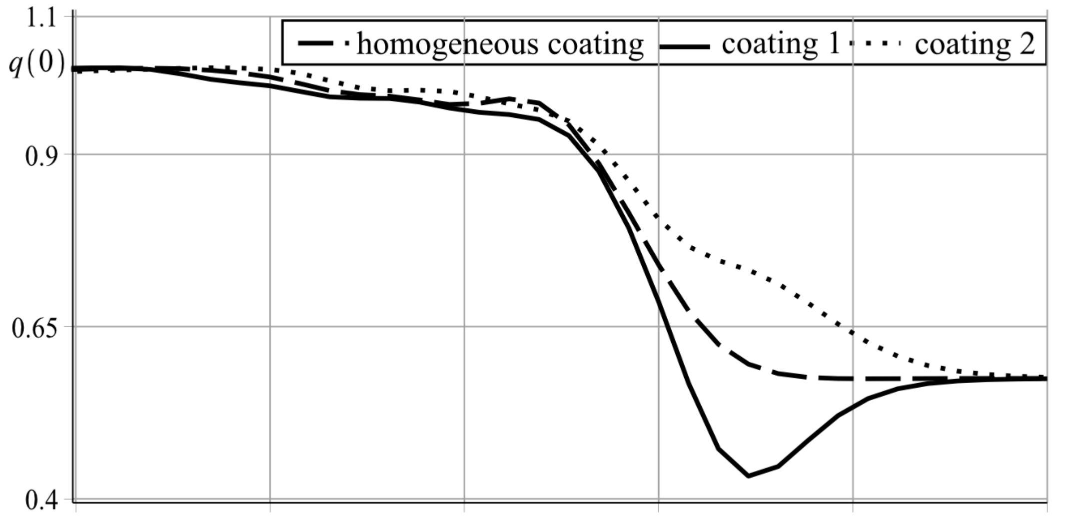

Figure 8. Graphs of pressure *q*(0) versus λ. The graphs are presented for soft homogeneous and functionally graded coatings with β = 100 and plate of intermediate flexibility s = 5.

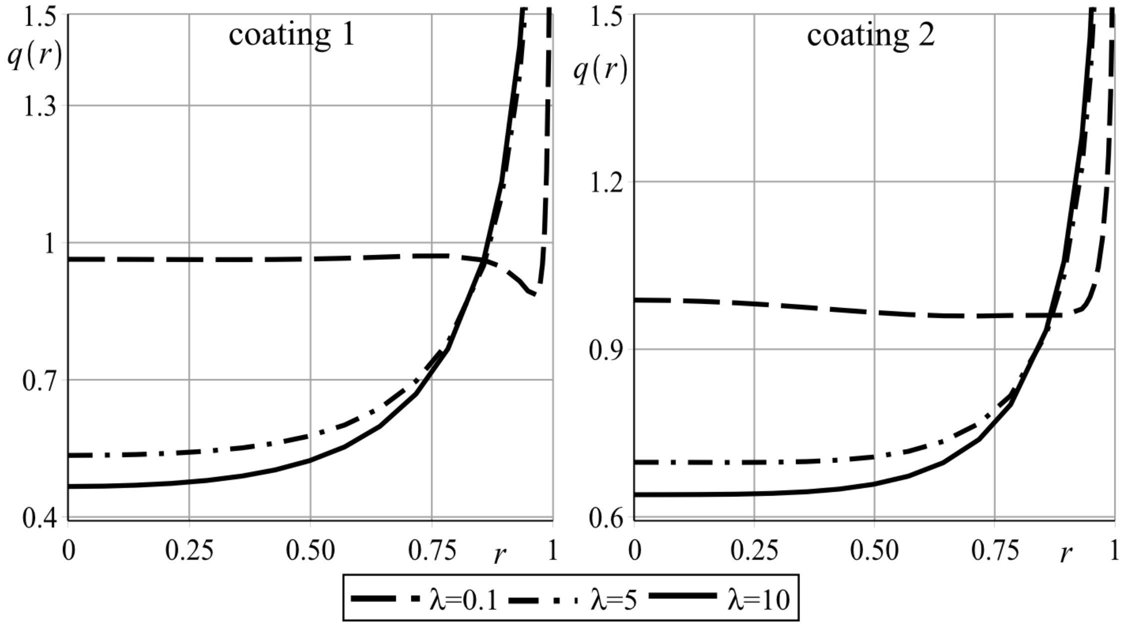

Figure 9. Graphs of pressure distribution *q*(*r*) versus *r*. The graphs are presented for soft coatings homogeneous and functionally graded coatings with β = 100, plate of intermediate flexibility s = 5 and some large and intermediate coating thicknesses λ.

3. CONCLUSIONS

Analytical expressions for the contact stresses appearing under the plate and the deflection function are constructed using coupled asymptotic method. The method allows to consider the elastic layer lying on a much stiffer substrate. Using approximations for the kernel transform of high accuracy it is possible to obtain a solution of the problem which is applicable for all possible values of λ and any stiffness of the plate. Same method was successfully applied to a wide class of contact problems for materials with functionally-graded coatings [9], [12], [16].

The authors acknowledge the support of the Russian Foundation for Basic Research (grants nos. 14-07-00705-a, 15-07-05820-a, 15-38-20790-mol_a_ved, 14-08-92003-NNS_a). S.M. Aizikovich also acknowledges support of the Ministry of Education and Science of Russia in the framework of Government Assignment.

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