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Effect of white layer on the distribution of stresses and strains in thin annular discs subject to internal pressure

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ABSTRACT

"White layer" is a term referring to hard layers of material in the vicinity of surfaces that are generated during various machining and deformation processes. The elastic modulus and yield stress within white layers may increase by 170% and 390%, respectively. It is therefore reasonable to expect that such a huge difference in the mechanical properties between the narrow surface layer and base material affects the distribution of stresses and strains. The present papers deals with thin annular discs subject to internal pressure. It is assumed that there is a narrow hard layer in the vicinity of the inner surface of the disc. The flow theory of plasticity based on the von Mises yield criterion and its associated flow rule are adopted in the plastic region of the disc. A semi-analytical solution is found.

Keywords: thin disk, plane stress conditions, von Mises yield criterion, flow theory of plasticity, residual stress and strain, semi-analytic solution.

1. Introduction

"White layer" is a term referring to hard layers of material in the vicinity of surfaces. Such layers are generated during various machining and deformation processes (Griffiths, 1987). The majority of publications devoted to white layers have been concerned with mechanisms of the generation of such layers and wear (Griffiths, 1987, Cho et al., 2012, Huang et al., 2013 among many others). Influence of white layers on the development of rolling contact fatigue has been demonstrated in Warren and Guo (2005) using a numerical method and in Choi (2010) using an experimental technique. Tribological advantages of white layers have been discussed in Griffiths and Furze (1987). It is of interest to understand how white layers affect structure and component performance under other loading conditions. In particular, it has been found in Cho et al. (2012) that the elastic modulus and yield stress within white layers may increase by 170% and 390%, respectively. It is therefore reasonable to expect that such a huge difference in the mechanical properties between the narrow surface layer and base material affects the distribution of stresses and strains including residual stresses and strains in structures under service conditions. Analytic and semi-analytic solutions are very useful to reveal this possible effect, even though such solutions by necessity involve simplifying assumptions. The present paper presents a semi-analytical solution for the elastic/plastic distribution stress and strain in a thin annular disc subject to pressure over its inner radius. It is assume that there is a white layer in the vicinity of the inner radius. The enlargement of a hole in plates or discs is one of the classical problems of plasticity. Solutions to this problem for various material models are contained in textbooks and monographs. A recent review of available solutions for the enlargement of a circular hole in thin plates has been given in Masri et al. (2010). However, most studies for elastic/plastic models have only focused on deformation theories of plasticity, unless Tresca's yield criterion is adopted. The present paper deals with the flow theory of plasticity and the von Mises yield criterion.

2. Statement of the problem

Consider a thin hollow disc of yield stress σ_0, Poisson's ratio ν, Young's modulus *E*, outer radius b_0, and inner radius a_0. It is assumed that there is a narrow hard layer in the vicinity of the inner surface of the disc. The yield stress and Young's modulus within this layer are denoted by σ_l and E_l, respectively, and the thickness of the layer by Δ (Figure 1). In general, $\sigma_l > \sigma_0$ and $E_l > E$. The disc is loaded by an uniform pressure, p_0, applied over its inner radius. The disc has no stress at $p_0 = 0$. Strains are supposed to be infinitesimal. At the stages of loading and unloading the state of stress is two-dimensional $(\sigma_z = 0)$ in a cylindrical coordinate system (r, θ, z) with its *z*-axis coinciding with the axis of symmetry of the disc. Here, σ_z is the axial stress (σ_r and σ_θ will stand for the radial and circumferential stresses, respectively). The boundary value problem is axisymmetric, and its solution is independent of θ. The circumferential displacement vanishes everywhere. The normal stresses in the cylindrical coordinate system are the principal stresses. The boundary conditions are

$$\sigma_r = 0 \tag{1}$$

for $r = b_0$ and

$$\sigma_r = -p_0 \tag{2}$$

for $r = a_0$.

It is assumed that the hard layer is purely elastic. The corresponding constitutive equations are

$$\varepsilon_r = \frac{\sigma_r - \nu\sigma_\theta}{E_l}, \quad \varepsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{E_l}, \quad \varepsilon_z = -\frac{\nu(\sigma_r + \sigma_\theta)}{E_l}. \tag{3}$$

Here ε_r, ε_θ and ε_z are the radial, circumferential ad axial strains, respectively, in the cylindrical coordinate system. The domain $a_0 + \Delta \le r \le b_0$ is in general elastic/plastic. In particular, if p_0 is high enough then this domain consists of two regions, elastic and plastic. The constitutive equations in the elastic region are

$$\varepsilon_r^e = \frac{\sigma_r - \nu\sigma_\theta}{E}, \quad \varepsilon_\theta^e = \frac{\sigma_\theta - \nu\sigma_r}{E}, \quad \varepsilon_z^e = -\frac{\nu\left(\sigma_r + \sigma_\theta\right)}{E}. \tag{4}$$

The superscript *e* denotes the elastic part of the strain and will denote the elastic part of the strain rate as well. In the elastic region, the whole strain is elastic. The superscript *e* is employed in Eq. (4) as the same equations are satisfied by the elastic part of the strain in the plastic region. The superscript can be dropped in the elastic region. It is assumed that the von Mises yield criterion and its associated flow rule are valid in the plastic region. These equations in plane stress are written as

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_r\sigma_\theta = \sigma_0^2 \tag{5}$$

and

$$\dot{\varepsilon}_r^p = \lambda\left(2\sigma_r - \sigma_\theta\right), \quad \dot{\varepsilon}_\theta^p = \lambda\left(2\sigma_\theta - \sigma_r\right), \quad \dot{\varepsilon}_z^p = -\lambda\left(\sigma_r + \sigma_\theta\right) \tag{6}$$

where λ is a non-negative multiplier. The superimposed dot denotes the time derivative at fixed *r*, and the superscript *p* denotes the plastic part of the strain and strain rate. Thus, $\dot{\varepsilon}_r^p$, $\dot{\varepsilon}_\theta^p$ and $\dot{\varepsilon}_z^p$ are the plastic strain rates. The total strains and strain rates in the plastic region are

$$\begin{aligned} &\varepsilon_r = \varepsilon_r^e + \varepsilon_r^p, \quad \varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p, \quad \varepsilon_z = \varepsilon_z^e + \varepsilon_z^p \\ &\dot{\varepsilon}_r = \dot{\varepsilon}_r^e + \dot{\varepsilon}_r^p, \quad \dot{\varepsilon}_\theta = \dot{\varepsilon}_\theta^e + \dot{\varepsilon}_\theta^p, \quad \dot{\varepsilon}_z = \dot{\varepsilon}_z^e + \dot{\varepsilon}_z^p. \end{aligned} \tag{7}$$

The material model adopted is rate-independent. Therefore, the time derivative can be replaced with the derivative with respect to any monotonically increasing or decreasing parameter *q*. In particular, it is convenient to introduce the following quantities

$$\begin{aligned} &\xi_r = \frac{\partial \varepsilon_r}{\partial q}, \quad \xi_\theta = \frac{\partial \varepsilon_\theta}{\partial q}, \quad \xi_z = \frac{\partial \varepsilon_z}{\partial q} \\ &\xi_r^e = \frac{\partial \varepsilon_r^e}{\partial q}, \quad \xi_\theta^e = \frac{\partial \varepsilon_\theta^e}{\partial q}, \quad \xi_z^e = \frac{\partial \varepsilon_z^e}{\partial q} \\ &\xi_r^p = \frac{\partial \varepsilon_r^p}{\partial q}, \quad \xi_\theta^e = \frac{\partial \varepsilon_\theta^p}{\partial q}, \quad \xi_z^e = \frac{\partial \varepsilon_z^p}{\partial q}. \end{aligned} \tag{8}$$

The constitutive equations should be supplemented with the equilibrium equation of the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{9}$$

It will be seen later that the use of the equation of strain rate compatibility facilitates the analysis. This equation is equivalent to

$$\frac{\partial \xi_\theta}{\partial r} + \frac{\xi_\theta - \xi_r}{r} = 0. \tag{10}$$

It is convenient to introduce the following dimensionless quantities

$$\rho = \frac{r}{b_0}, \quad a = \frac{a_0}{b_0}, \quad \delta = \frac{\Delta}{b_0}, \quad k = \frac{\sigma_0}{E}, \quad k_l = \frac{\sigma_l}{E_l}, \quad s = \frac{\sigma_l}{\sigma_0}, \quad p = \frac{p_0}{\sigma_0}. \tag{11}$$

3. Purely elastic solution in the hard layer

The general axisymmetric purely elastic solution under plane stress conditions is well known (see, for example, Hill (1950)). Using Eq. (11) this solution can be written as

$$\begin{gathered} \frac{\sigma_r}{\sigma_0} = \frac{A_l}{\rho^2} + B_l, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{A_l}{\rho^2} + B_l, \\ \frac{s\varepsilon_r}{k_l} = \frac{A_l(1+\nu)}{\rho^2} + B_l(1-\nu), \quad \frac{s\varepsilon_\theta}{k_l} = -\frac{A_l(1+\nu)}{\rho^2} + B_l(1-\nu), \quad \frac{s\varepsilon_z}{k_l} = -2\nu B_l. \end{gathered} \tag{12}$$

Here A_l and B_l are constants of integration. Using Eq. (11) the boundary condition (2) and the solution (12) combine to give

$$B_l = -p - \frac{A_l}{a^2}. \tag{13}$$

Then, it follows from Eqs. (12) and (13) that

$$\frac{\sigma_\delta}{\sigma_0} = A_l\left[\frac{1}{(a+\delta)^2} - \frac{1}{a^2}\right] - p, \quad \frac{s\varepsilon_\delta}{k_l} = -A_l\left[\frac{(1+\nu)}{(a+\delta)^2} + \frac{(1-\nu)}{a^2}\right] - p(1-\nu). \tag{14}$$

Here σ_δ and ε_δ are the values of σ_r and ε_θ, respectively, at $r = a_0 + \Delta$ (or $\rho = a + \delta$). Differentiating Eq. (14) for ε_δ with respect to q leads to

$$\frac{s\xi_\delta}{k_l} = -\left[\frac{(1+\nu)}{(a+\delta)^2} + \frac{(1-\nu)}{a^2}\right]\frac{dA_l}{dq} - (1-\nu)\frac{dp}{dq}. \tag{15}$$

Here ξ_δ is the value of ξ_θ at $\rho = a + \delta$.

4. Purely elastic solution in the domain $a + \delta \le \rho \le 1$ and the initiation of plastic yielding

The purely elastic solution in the domain $a + \delta \le \rho \le 1$ has the same form as Eq. (12). In particular, using Eq. (11)

$$\begin{gathered} \frac{\sigma_r}{\sigma_0} = \frac{A}{\rho^2} + B, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{A}{\rho^2} + B, \\ \frac{\varepsilon_r}{k} = \frac{A(1+\nu)}{\rho^2} + B(1-\nu), \quad \frac{\varepsilon_\theta}{k} = -\frac{A(1+\nu)}{\rho^2} + B(1-\nu), \quad \frac{\varepsilon_z}{k} = -2\nu B. \end{gathered}$$

(16)

Here A and B are new constants of integration. Using the boundary condition (1) yields

$$A + B = 0. \tag{17}$$

The radial stress and the circumferential strain are continuous across the surface $\rho = a + \delta$. Therefore, it follows from Eqs. (14), (16) and (17) that

$$\begin{gathered}A\left[\frac{1}{(a+\delta)^2} - 1\right] - A_l\left[\frac{1}{(a+\delta)^2} - \frac{1}{a^2}\right] = -p,\\ -A\left[1 - \nu + \frac{(1+\nu)}{(a+\delta)^2}\right] + A_l\left[\frac{(1+\nu)}{(a+\delta)^2} + \frac{(1-\nu)}{a^2}\right]\frac{k_l}{sk} = -\frac{p(1-\nu)k_l}{sk}.\end{gathered} \tag{18}$$

This is a linear system of equations for A and A_l. Its solution can be readily found. In particular,

$$\begin{gathered}A = -2p\left\{\frac{ks}{k_l}\left[\frac{(a+\delta)^2}{a^2} - 1\right]\left[1 - \nu + \frac{(1+\nu)}{(a+\delta)^2}\right] + \left[\frac{(1-\nu)}{a^2} + \frac{(1+\nu)}{(a+\delta)^2}\right]\left[1 - (a+\delta)^2\right]\right\}^{-1},\\ A_l = p\left\{(1-\nu)\left[(a+\delta)^2 - 1\right] - \frac{ks}{k_l}\left[(1-\nu)(a+\delta)^2 + 1 + \nu\right]\right\}\times\\ \left\{\frac{ks}{k_l}\left[\frac{(a+\delta)^2}{a^2} - 1\right]\left[1 - \nu + \frac{(1+\nu)}{(a+\delta)^2}\right] + \left[\frac{(1-\nu)}{a^2} + \frac{(1+\nu)}{(a+\delta)^2}\right]\left[1 - (a+\delta)^2\right]\right\}^{-1}.\end{gathered} \tag{19}$$

Using Eqs. (13) and (19) the distribution of stresses and strains within the hard layer is determined from Eq. (12). Analogously, using Eqs. (17) and (19) the distribution of stresses and strains in the range $a + \delta \le \rho \le 1$ is determined from Eq. (16). Substituting Eqs. (16) and (17) at $\rho = a + \delta$ into Eq. (5) yields

$$A_e = -\frac{(a+\delta)^2}{\sqrt{1 + 3(a+\delta)^4}}. \tag{20}$$

Here A_e is the value of A at which the plastic region starts to develop at $\rho = a + \delta$. It follows from Eqs. (19) and (20) that

$$p_e = \frac{(a+\delta)^2}{2\sqrt{1 + 3(a+\delta)^4}}\left\{\begin{gathered}\frac{ks}{k_l}\left[\frac{(a+\delta)^2}{a^2} - 1\right]\left[1 - \nu + \frac{(1+\nu)}{(a+\delta)^2}\right] +\\ \left[\frac{(1-\nu)}{a^2} + \frac{(1+\nu)}{(a+\delta)^2}\right]\left[1 - (a+\delta)^2\right]\end{gathered}\right\}. \tag{21}$$

Here p_e is the value of *p* at which the plastic region starts to develop at $\rho = a + \delta$. In what follows, it is assumed that $p \geq p_e$.

5. Elastic/plastic solution in the domain $a + \delta \leq \rho \leq 1$

Let $\rho = \rho_c$ be the elastic/plastic boundary. The general solution (16) is valid in the elastic region $\rho_c \leq \rho \leq 1$. Equation (17) is also valid. However, *A* and A_l are not determined from Eq. (19). In the plastic region, $a + \delta \leq \rho \leq \rho_c$, there are two stress equations, Eqs. (5) and (9). The yield criterion (5) is satisfied by the following standard substitution

$$\frac{\sigma_r}{\sigma_0} = -\frac{2\sin\psi}{\sqrt{3}}, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{\sin\psi}{\sqrt{3}} - \cos\psi. \tag{22}$$

Substituting this equation into Eq. (9) ad using Eq. (11) yield

$$2\rho\cos\psi\frac{\partial\psi}{\partial\rho} = \sqrt{3}\cos\psi - \sin\psi. \tag{23}$$

It follows from Eqs. (14) and (22) that

$$\frac{\sigma_\delta}{\sigma_0} = A_l\left[\frac{1}{(a+\delta)^2} - \frac{1}{a^2}\right] - p = -\frac{2\sin\psi_\delta}{\sqrt{3}}. \tag{24}$$

Therefore, the boundary condition to Eq. (23) is

$$\psi = \psi_\delta \tag{25}$$

for $\rho = a + \delta$. The solution of Eq. (23) satisfying this boundary condition is

$$\ln\left(\frac{\rho}{a+\delta}\right) = 2\int_{\psi_\delta}^{\psi} \frac{\cos\chi}{\left(\sqrt{3}\cos\chi - \sin\chi\right)} d\chi \tag{26}$$

where χ is a dummy variable of integration. The integral in Eq. (26) can be evaluated to give

$$\rho = (a+\delta)\exp\left[\frac{\sqrt{3}}{2}(\psi - \psi_\delta)\right]\sqrt{\frac{\sin(\psi_\delta - \pi/3)}{\sin(\psi - \pi/3)}}. \tag{27}$$

Using Eq. (11) the elastic strain in the plastic region are determined from Eqs. (4) and (22) as

$$\frac{\varepsilon_r^e}{k} = \nu\cos\psi - \frac{(2-\nu)}{\sqrt{3}}\sin\psi, \quad \frac{\varepsilon_\theta^e}{k} = -\cos\psi - \frac{(1-2\nu)}{\sqrt{3}}\sin\psi, \quad \frac{\varepsilon_z^e}{k} = 2\nu\cos\left(\psi - \frac{\pi}{3}\right). \tag{28}$$

In what follows, it is assumed that $q \equiv \psi_\delta$. Then, differentiating Eq. (28) with respect to ψ_δ leads to

$$\frac{\xi_r^e}{k} = -\left[\nu \sin\psi + \frac{(2-\nu)}{\sqrt{3}}\cos\psi\right]\frac{\partial\psi}{\partial\psi_\delta}, \quad \frac{\xi_\theta^e}{k} = \left[\sin\psi - \frac{(1-2\nu)}{\sqrt{3}}\cos\psi\right]\frac{\partial\psi}{\partial\psi_\delta},$$
$$\frac{\xi_z^e}{k} = -2\nu\sin\left(\psi - \frac{\pi}{3}\right)\frac{\partial\psi}{\partial\psi_\delta}. \tag{29}$$

In order to find the derivative $\partial\psi/\partial\psi_\delta$, it is necessary to differentiate Eq. (26). As a result,

$$\frac{d\rho}{\rho} = \frac{2\cos\psi d\psi}{\left(\sqrt{3}\cos\psi - \sin\psi\right)} - \frac{2\cos\psi_\delta d\psi_\delta}{\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)}. \tag{30}$$

It follows from this equation that

$$\frac{\partial\psi}{\partial\psi_\delta} = \frac{\left(\sqrt{3}\cos\psi - \sin\psi\right)\cos\psi_\delta}{\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)\cos\psi}.$$
(31)

Eliminating λ in Eq. (6) and replacing the time derivative with the derivative with respect to ψ_δ yield

$$\xi_r^p = \xi_\theta^p \frac{(2\sigma_r - \sigma_\theta)}{(2\sigma_\theta - \sigma_r)}, \quad \xi_z^p = -\xi_\theta^p \frac{(\sigma_r + \sigma_\theta)}{(2\sigma_\theta - \sigma_r)}. \tag{32}$$

Eliminating σ_r and σ_θ in these equations by means of Eq. (22) gives

$$\xi_r^p = \xi_\theta^p \frac{\sin(\psi - \pi/6)}{\cos\psi}, \quad \xi_z^p = -\xi_\theta^p \frac{\sin(\psi + \pi/6)}{\cos\psi}. \tag{33}$$

Using Eqs. (7), (11) and (33), Eq. (10) can be rewritten as

$$\rho\frac{\partial\xi_\theta}{\partial\rho} = \xi_r^e - \xi_\theta^e - \frac{\sqrt{3}}{2}\left(\sqrt{3} - \tan\psi\right)\left(\xi_\theta - \xi_\theta^e\right).$$

Replacing here differentiation with respect to ρ with differentiation with respect to ψ by means of Eq. (23) leads to

$$\frac{\partial\xi_\theta}{\partial\psi} + \sqrt{3}\xi_\theta = \frac{2\left(\xi_r^e - \xi_\theta^e\right)}{\left(\sqrt{3} - \tan\psi\right)} + \sqrt{3}\xi_\theta^e. \tag{34}$$

Using Eqs. (29) and (31) it is possible to represent ξ_r^e and ξ_θ^e as functions of ψ and ψ_δ. Then, Eq. (34) becomes

$$\frac{\partial \xi_\theta}{\partial \psi} + \sqrt{3}\xi_\theta = \frac{k\cos\psi_\delta}{\sqrt{3}\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)} \frac{\left[(1-2\nu)\left(\sqrt{3}\sin 2\psi - \cos 2\psi\right) - 2(2-\nu)\right]}{\cos\psi}. \qquad (35)$$

Let ξ_c be the value of ξ_θ at $\rho = \rho_c$ and ψ_c be the value of ψ at $\rho = \rho_c$. The solution of equation (35) satisfying the boundary condition $\xi_\theta = \xi_c$ at $\psi = \psi_c$ is

$$\frac{\xi_\theta}{k} = \frac{\xi_c}{k}\exp\left[\sqrt{3}(\psi_c - \psi)\right] + \frac{\cos\psi_\delta}{\sqrt{3}\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)} \times \int_{\psi_c}^{\psi} \frac{\left[(1-2\nu)\left(\sqrt{3}\sin 2\chi - \cos 2\chi\right) - 2(2-\nu)\right]}{\cos\chi} \exp\left[\sqrt{3}(\chi - \psi)\right] d\chi. \qquad (36)$$

The equation for the strain ε_θ is

$$\frac{\partial \varepsilon_\theta}{\partial \psi_\delta} + \frac{\partial \varepsilon_\theta}{\partial \psi}\frac{\partial \psi}{\partial \psi_\delta} = \xi_\theta.$$

Using Eq. (31) this equation ca be rewritten as

$$\frac{\partial \varepsilon_\theta}{\partial \psi_\delta}\frac{\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)}{\cos\psi_\delta} + \frac{\partial \varepsilon_\theta}{\partial \psi}\frac{\left(\sqrt{3}\cos\psi - \sin\psi\right)}{\cos\psi} = \frac{\xi_\theta\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)}{\cos\psi_\delta}. \qquad (37)$$

It is evident from this equation that $\psi = \psi_\delta$ is a characteristic curve and the relation along this characteristic curve is

$$\frac{d\xi_\theta}{d\psi_\delta} = \xi_\delta. \qquad (38)$$

Here ξ_δ is given by Eq. (15) in which differentiation in terms of *q* should be replaced with differentiation with respect to ψ_δ. On the other hand, it follows from Eq. (36) that

$$\frac{\xi_\delta}{k} = \frac{\xi_c}{k}\exp\left[\sqrt{3}(\psi_c - \psi_\delta)\right] + \frac{\cos\psi_\delta}{\sqrt{3}\left(\sqrt{3}\cos\psi_\delta - \sin\psi_\delta\right)} \times \int_{\psi_c}^{\psi_\delta} \frac{\left[(1-2\nu)\left(\sqrt{3}\sin 2\chi - \cos 2\chi\right) - 2(2-\nu)\right]}{\cos\chi} \exp\left[\sqrt{3}(\chi - \psi_\delta)\right] d\chi. \qquad (39)$$

In order to solve Eq. (38), it is necessary to find ψ_c and ξ_c as functions of ψ_δ. The stresses σ_r and σ_θ are continuous across the elastic/plastic boundary. Therefore, it follows from Eqs. (16), (17) and (22) that

$$\frac{2\sin\psi_c}{\sqrt{3}} = A\left(1-\frac{1}{\rho_c^2}\right), \quad \frac{\sin\psi_c}{\sqrt{3}} + \cos\psi_c = A\left(1+\frac{1}{\rho_c^2}\right).$$

Solving these equations for ρ_c and *A* gives

$$\rho_c^2 = -\frac{\sqrt{3}\sin(\psi_c+\pi/6)}{\sin(\psi_c-\pi/3)}, \quad A = \frac{1}{2}\left(\sqrt{3}\sin\psi_c + \cos\psi_c\right). \tag{40}$$

Equation (27) supplies the relation

$$\rho_c = (a+\delta)\exp\left[\frac{\sqrt{3}}{2}(\psi_c - \psi_\delta)\right]\sqrt{\frac{\sin(\psi_\delta-\pi/3)}{\sin(\psi_c-\pi/3)}}. \tag{41}$$

The relation that connects ψ_c and ψ_δ is readily determined from Eqs. (40) and (41) as

$$(a+\delta)^2\exp\left[\sqrt{3}(\psi_c-\psi_\delta)\right] + \frac{\sqrt{3}\sin(\psi_c+\pi/6)}{\sin(\psi_\delta-\pi/3)} = 0. \tag{42}$$

Both ε_θ and ξ_θ are continuous across the elastic/plastic boundary. Therefore, ξ_c is equal to the value of ξ_θ on the elastic side of this boundary. The latter is found from Eq. (16). Then, using Eqs. (17) and (40)

$$\frac{\xi_c}{k} = \frac{1}{2}\left[\frac{(1+\nu)}{\rho_c^2} + 1 - \nu\right]\left(\sin\psi_c - \sqrt{3}\cos\psi_c\right)\frac{d\psi_c}{d\psi_\delta}. \tag{43}$$

Differentiating Eq. (42) and using Eqs. (40) and (41) give

$$\frac{d\psi_c}{d\psi_\delta} = \frac{\sin(\psi_c+\pi/6)}{2\sin\psi_c}\left[\sqrt{3} - \cot\left(\psi_\delta - \frac{\pi}{3}\right)\right]. \tag{44}$$

Substituting Eqs. (40) and (44) into Eq. (43) results in

$$\frac{\xi_c}{k} = -\frac{1}{8}\left[1-\nu+\frac{(1+\nu)}{\sqrt{3}}\cot\left(\psi_c+\frac{\pi}{6}\right)\right]\left[\sqrt{3}-\cot\left(\psi_\delta-\frac{\pi}{3}\right)\right]\frac{\left(\sqrt{3}\cos 2\psi_c + \sin 2\psi_c\right)}{\sin\psi_c}. \tag{45}$$

Equation (42) should be solved numerically for ψ_c. Having found the value of ψ_c, it is possible to eliminate ξ_c in Eq. (39) by means of Eq. (45). This determines the right hand side of Eq. (38) as a function of ψ_a. Subsequent numerical integration supplies the value of ε_θ. The other strains can be found is a similar manner.

6. Conclusions

A new semi-analytical solution for the elastic/plastic distribution of stresses and strains in a thin annular disk subject to pressure over the edge of the hole has been found. It is assumed that there is a white layer in the vicinity of the edge of the hole. This layer is purely elastic. The constitutive equations are the von Mises yield criterion and its associated flow rule. Thus the material is plastically incompressible. The boundary value problem has been reduced to solving transcendental equations and numerical integration.

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